Zeta-functions of weight lattices of compact connected semisimple Lie groups

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§1. Introduction (Background)

Zeta-functions of Lie groups = constant part of

partition functions of quantum gauge theories.

§§1.1.Paritition Functions

Partition function = "generating function" of probabilities.

example: Consider a die (a small cube whose faces are numbered from 1 to 6) and the probability of throwing each number n ($n \in \{1,\dots,6\}$) is proportional to w_n.

the average of numbers in one throw: $$\frac{\sum_{n=1}^{6} n w_n}{\sum_{n=1}^{6} w_n}.$$

Partition function:

$$Z(\tau) = \sum_{n=1}^{6} w_n e^{-\tau n},$$

$$-\frac{d}{d\tau}(\log Z(\tau))\Big|_{\tau=0} = -\frac{Z'(0)}{Z(0)} = \frac{\sum_{n=1}^{6} n w_n}{\sum_{n=1}^{6} w_n}.$$

- By higher order derivatives of $Z(\tau)$, the avarages of n^2, n^3, $\dots$, (moments) and $f(n)$ can be calculated.
- In particular, if $w_n = 1$, then $Z(0) = \sharp$states $=$ volume.

§§1.2. Quantum Gauge Theory

Σ: orientable closed 2-dimensional surface. A: connection of a principal G-bundle. F: curvature.

Paritition funciton: ($\int \mathcal{D}A$ denotes the path integral.)

$$Z_{\Sigma;G}(\tau) = \int \mathcal{D}A\, e^{-I(A)}, \qquad I(A) = \frac{1}{\tau}\int_\Sigma d\mu \|F(A)\|^2, \qquad F(A) = dA + \frac{1}{2}[A \wedge A].$$

In particular, when $\tau \to 0$, only flat connections A (i.e., $\|F(A)\| = 0$) contributes to $Z_{\Sigma;G}(\tau)$.

$$Z_{\Sigma;G}(0) = \int_{\text{flat connections}} 1 \quad \mathcal{D}A = \text{“the volume of flat connections”}.$$

Witten:

$$Z_{\Sigma;G}(\tau) = \sum_\varphi \frac{e^{-\tau c_2(\varphi)/2}}{(\dim \varphi)^{2g-2}}. \qquad \text{(theta \& zeta)}$$

g: genus of Σ. φ: all finite dimensional irreducible representations of G. $c_2(\varphi)$: Casimir values.

Prototype of Zeta-functions of Lie groups (Our starting point).

$$Z_{\Sigma;G}(0) = \sum_\varphi \frac{1}{(\dim \varphi)^{2g-2}} = \text{“the volume of flat connections”}.$$

eg. $\displaystyle\sum_{m=1}^\infty \frac{1}{m^s} = \zeta(s), \qquad \sum_{m,n=1}^\infty \frac{2^s}{m^s n^s (m+n)^s}, \qquad \sum_{m,n=1}^\infty \frac{6^s}{m^s n^s (m+n)^s (m+2n)^s} \qquad (s = 2g-2)$

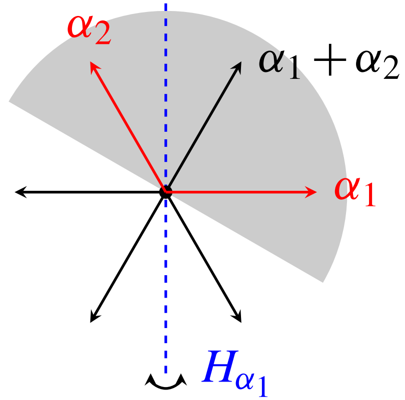
§2. Root Systems

§§2.1. Definitions

Let V be an r dimensional real vector space equipped with inner product $\langle\cdot,\cdot\rangle$.

A root system $\Delta \subset V$ is a set of vectors (roots):

1. finiteness: $|\Delta| < \infty$ and $0 \notin \Delta$.
2. symmetry: $\sigma_\alpha \Delta = \Delta$ for all $\alpha \in \Delta$.
3. integrality: $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$.
4. $\alpha, c\alpha \in \Delta \Longrightarrow c = \pm 1$.



$|\Delta| = 6.$

$W \simeq \mathfrak{S}_3.$

$|W| = 6.$

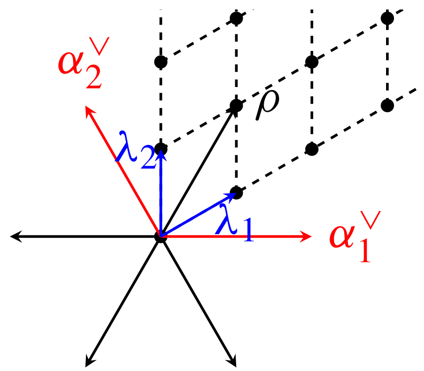
σ_α: the reflection with respect to the hyperplane H_α orthogonal to α.

W: Weyl group (the group generated by all σ_α.)

$\alpha^\vee$: coroot $(= 2\alpha/\langle\alpha,\alpha\rangle.\ \alpha^{\vee\vee} = \alpha.)$

Δ_+: positive roots (all roots contained in a fixed half space.)

$\{\alpha_1,\ldots,\alpha_r\}$: fundamental roots (a basis s.t. $\forall\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta_+$ with all $c_i \geq 0$.)



P_+: dominant weights $(= \bigoplus \mathbb{Z}_{\geq 0}\lambda_i$, $\{\lambda_1,\ldots,\lambda_r\}$ dual basis of $\{\alpha_1^\vee,\ldots,\alpha_r^\vee\}$.) ρ: Weyl vector $(= \frac{1}{2}\sum \lambda_i$.)

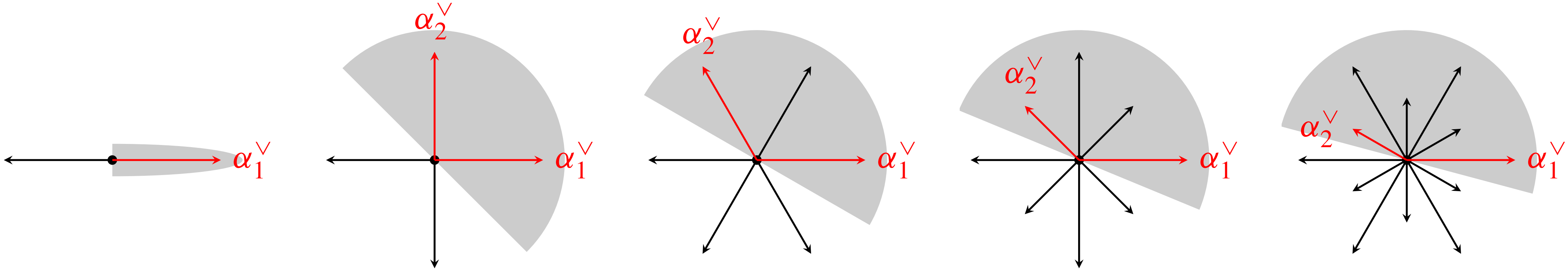
Remember: A nice group W acts on Δ.

§§2.2. Examples

In the following, we mainly use $\Delta^\vee$ instead of Δ. (Note Δ is a root system $\Leftrightarrow$ $\Delta^\vee$ is a root system.)

§§§2.2.1. Rank 1 and 2

A_1 | $A_1 \times A_1$ | A_2 | C_2 (or B_2) | G_2



$\Delta_+^\vee = \{\alpha_1^\vee\}$

$\{\alpha_1^\vee, \alpha_2^\vee\}$

$\left\{ \begin{matrix} \alpha_1^\vee, \alpha_2^\vee \\ \alpha_1^\vee + \alpha_2^\vee \end{matrix} \right\}$

$\left\{ \begin{matrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \end{matrix} \right\}$

$\left\{ \begin{matrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \\ \alpha_1^\vee + 3\alpha_2^\vee \\ 2\alpha_1^\vee + 3\alpha_2^\vee \end{matrix} \right\}$

§§§2.2.2. Higher Rank

Root systems are classified as $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

§3. Zeta-Functions of Root Systems

§§3.1. Witten Zeta-Functions

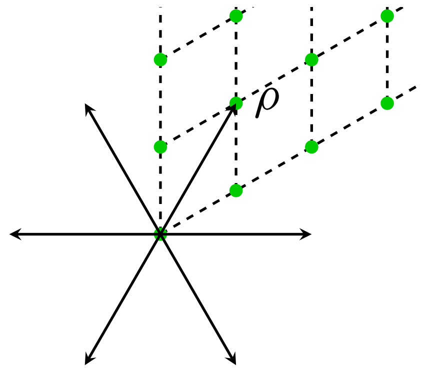
Known facts: $\varphi \iff \lambda \in P_+$, $\quad \dim\varphi = \prod_{\alpha\in\Delta_+} \frac{\langle\alpha^\vee, \lambda+\rho\rangle}{\langle\alpha^\vee, \rho\rangle}$ (Weyl's dimension formula).

Witten zeta-functions (Witten 1991, Zagier 1994):

For a complex simple Lie algebra $\mathfrak{g}$ of type X_r,

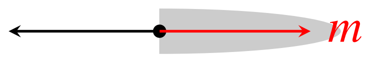
$$\zeta_W(s;X_r) = \sum_{\varphi} \frac{1}{(\dim\varphi)^s} = K(X_r)^s \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee, \lambda+\rho\rangle^s},$$

where the summation runs over all finite dimensional irreducible representations φ.

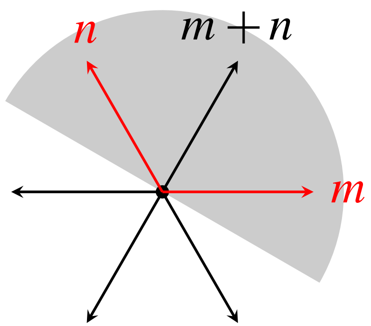


Formally replace $\alpha_1^\vee$ and $\alpha_2^\vee$ by m and n appearing in positive coroots.

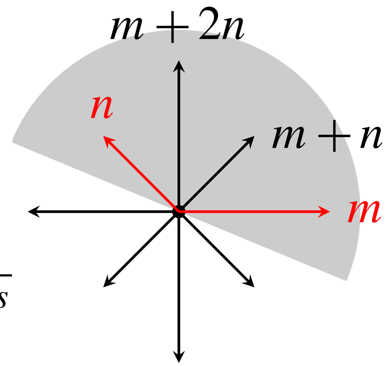
$$\zeta_W(s;A_1) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s)$$



$$\zeta_W(s;A_2) = 2^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s}$$



$$\zeta_W(s;C_2) = 6^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}$$



§§3.2. Example: E_8 Case

$$\zeta_W(s;E_8) = K(E_8)^s \sum_{m_1,m_2,m_3,m_4,m_5,m_6,m_7,m_8=1}^{\infty} \Big(m_1m_2m_3m_4m_5m_6m_7m_8(m_1+m_3)(m_2+m_4)(m_3+m_4)(m_4+m_5)(m_5+m_6)(m_6+m_7)(m_7+m_8)(m_1+m_3+m_4)(m_2+m_3+m_4)(m_2+m_4+m_5)(m_3+m_4+m_5)(m_4+m_5+m_6)(m_5+m_6+m_7)(m_6+m_7+m_8)(m_1+m_2+m_3+m_4)(m_1+m_3+m_4+m_5)(m_2+m_3+m_4+m_5)(m_2+m_4+m_5+m_6)(m_3+m_4+m_5+m_6)(m_4+m_5+m_6+m_7)(m_5+m_6+m_7+m_8)(m_1+m_2+m_3+m_4+m_5)(m_1+m_3+m_4+m_5+m_6)(m_2+m_3+2m_4+m_5)(m_2+m_3+m_4+m_5+m_6)(m_2+m_4+m_5+m_6+m_7)(m_3+m_4+m_5+m_6+m_7)(m_4+m_5+m_6+m_7+m_8)(m_1+m_2+m_3+2m_4+m_5)(m_1+m_2+m_3+m_4+m_5+m_6)(m_1+m_3+m_4+m_5+m_6+m_7)(m_2+m_3+2m_4+m_5+m_6)(m_2+m_3+m_4+m_5+m_6+m_7)(m_2+m_4+m_5+m_6+m_7+m_8)(m_3+m_4+m_5+m_6+m_7+m_8)(m_1+m_2+2m_3+2m_4+m_5)(m_1+m_2+m_3+2m_4+m_5+m_6)(m_1+m_2+m_3+m_4+m_5+m_6+m_7)(m_1+m_3+m_4+m_5+m_6+m_7+m_8)(m_2+m_3+2m_4+2m_5+m_6)(m_2+m_3+2m_4+m_5+m_6+m_7)(m_2+m_3+m_4+m_5+m_6+m_7+m_8)(m_1+m_2+2m_3+2m_4+m_5+m_6)(m_1+m_2+m_3+2m_4+2m_5+m_6)(m_1+m_2+m_3+2m_4+m_5+m_6+m_7)(m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8)(m_2+m_3+2m_4+2m_5+m_6+m_7)(m_2+m_3+2m_4+m_5+m_6+m_7+m_8)(m_1+m_2+2m_3+2m_4+2m_5+m_6)(m_1+m_2+2m_3+2m_4+m_5+m_6+m_7)(m_1+m_2+m_3+2m_4+2m_5+m_6+m_7)(m_1+m_2+m_3+2m_4+m_5+m_6+m_7+m_8)(m_2+m_3+2m_4+2m_5+2m_6+m_7)(m_2+m_3+2m_4+2m_5+m_6+m_7+m_8)(m_1+m_2+2m_3+3m_4+2m_5+m_6)(m_1+m_2+2m_3+2m_4+2m_5+m_6+m_7)(m_1+m_2+2m_3+2m_4+m_5+m_6+m_7+m_8)(m_1+m_2+m_3+2m_4+2m_5+2m_6+m_7)(m_1+m_2+m_3+2m_4+2m_5+m_6+m_7+m_8)(m_2+m_3+2m_4+2m_5+2m_6+m_7+m_8)(m_1+2m_2+2m_3+3m_4+2m_5+m_6)(m_1+m_2+2m_3+3m_4+2m_5+m_6+m_7)(m_1+m_2+2m_3+2m_4+2m_5+2m_6+m_7)(m_1+m_2+2m_3+2m_4+2m_5+m_6+m_7+m_8)(m_1+m_2+m_3+2m_4+2m_5+2m_6+m_7+m_8)(m_2+m_3+2m_4+2m_5+2m_6+2m_7+m_8)(m_1+2m_2+2m_3+3m_4+2m_5+m_6+m_7)(m_1+m_2+2m_3+3m_4+2m_5+2m_6+m_7)(m_1+m_2+2m_3+3m_4+2m_5+m_6+m_7+m_8)(m_1+m_2+2m_3+2m_4+2m_5+2m_6+m_7+m_8)(m_1+m_2+m_3+2m_4+2m_5+2m_6+2m_7+m_8)(m_1+2m_2+2m_3+3m_4+2m_5+2m_6+m_7)(m_1+2m_2+2m_3+3m_4+2m_5+m_6+m_7+m_8)(m_1+m_2+2m_3+3m_4+3m_5+2m_6+m_7)(m_1+m_2+2m_3+3m_4+2m_5+2m_6+m_7+m_8)(m_1+m_2+2m_3+2m_4+2m_5+2m_6+2m_7+m_8)(m_1+2m_2+2m_3+3m_4+3m_5+2m_6+m_7)(m_1+2m_2+2m_3+3m_4+2m_5+2m_6+m_7+m_8)(m_1+m_2+2m_3+3m_4+3m_5+2m_6+m_7+m_8)(m_1+m_2+2m_3+3m_4+2m_5+2m_6+2m_7+m_8)(m_1+2m_2+2m_3+4m_4+3m_5+2m_6+m_7)(m_1+2m_2+2m_3+3m_4+3m_5+2m_6+m_7+m_8)(m_1+2m_2+2m_3+3m_4+2m_5+2m_6+2m_7+m_8)(m_1+m_2+2m_3+3m_4+3m_5+2m_6+2m_7+m_8)(m_1+2m_2+3m_3+4m_4+3m_5+2m_6+m_7)(m_1+2m_2+2m_3+4m_4+3m_5+2m_6+m_7+m_8)(m_1+2m_2+2m_3+3m_4+3m_5+2m_6+2m_7+m_8)(m_1+m_2+2m_3+3m_4+3m_5+3m_6+2m_7+m_8)(2m_1+2m_2+3m_3+4m_4+3m_5+2m_6+m_7)(m_1+2m_2+3m_3+4m_4+3m_5+2m_6+m_7+m_8)(m_1+2m_2+2m_3+4m_4+3m_5+2m_6+2m_7+m_8)(m_1+2m_2+2m_3+3m_4+3m_5+3m_6+2m_7+m_8)(2m_1+2m_2+3m_3+4m_4+3m_5+2m_6+m_7+m_8)(m_1+2m_2+3m_3+4m_4+3m_5+2m_6+2m_7+m_8)(m_1+2m_2+2m_3+4m_4+3m_5+3m_6+2m_7+m_8)(2m_1+2m_2+3m_3+4m_4+3m_5+2m_6+2m_7+m_8)(m_1+2m_2+3m_3+4m_4+3m_5+3m_6+2m_7+m_8)(m_1+2m_2+2m_3+4m_4+4m_5+3m_6+2m_7+m_8)(2m_1+2m_2+3m_3+4m_4+3m_5+3m_6+2m_7+m_8)(m_1+2m_2+3m_3+4m_4+4m_5+3m_6+2m_7+m_8)(2m_1+2m_2+3m_3+4m_4+4m_5+3m_6+2m_7+m_8)(m_1+2m_2+3m_3+5m_4+4m_5+3m_6+2m_7+m_8)(2m_1+2m_2+3m_3+5m_4+4m_5+3m_6+2m_7+m_8)(m_1+3m_2+3m_3+5m_4+4m_5+3m_6+2m_7+m_8)(2m_1+3m_2+3m_3+5m_4+4m_5+3m_6+2m_7+m_8)(2m_1+2m_2+4m_3+5m_4+4m_5+3m_6+2m_7+m_8)(2m_1+3m_2+4m_3+5m_4+4m_5+3m_6+2m_7+m_8)(2m_1+3m_2+4m_3+6m_4+4m_5+3m_6+2m_7+m_8)(2m_1+3m_2+4m_3+6m_4+5m_5+3m_6+2m_7+m_8)(2m_1+3m_2+4m_3+6m_4+5m_5+4m_6+2m_7+m_8)(2m_1+3m_2+4m_3+6m_4+5m_5+4m_6+3m_7+m_8)(2m_1+3m_2+4m_3+6m_4+5m_5+4m_6+3m_7+2m_8)\Big)^{-s}.$$

120 factors, $|W| = 696,729,600$ (large symmetry).

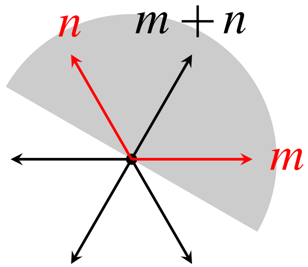
Classification: $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

§§3.3. Zeta-Functions of Root Systems

Definition 1 (Zeta-functions of root systems (KMT 2008, multivariable Lerch analog)). For a root system Δ of type X_r and for $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$ and $\mathbf{y} \in V$, define

$$\zeta_r(\mathbf{s},\mathbf{y};X_r) = \sum_{\lambda\in P_+} e^{2\pi i\langle \mathbf{y},\lambda+\rho\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee,\lambda+\rho\rangle^{s_\alpha}},$$

eg. $\zeta_2((s_1,s_2,s_3),(y_1,y_2);A_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i(my_1+ny_2)}}{m^{s_1}n^{s_2}(m+n)^{s_3}}$



Extend $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+}$ to $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha = s_{-\alpha}$ and define $(w\mathbf{s})_\alpha = s_{w^{-1}\alpha}$.

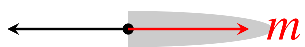
Theorem 1 (KMT 2008). *For* $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Big)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};X_r) = (-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Big),$$

where $P(\mathbf{k},\mathbf{y};X_r)$ *is a multiple periodic Bernoulli function (defined later).*

cf. $(X_r = A_1)$

$$\varphi(s,y) = \sum_{m=1}^{\infty}\frac{e^{2\pi i my}}{m^s},$$



$$\varphi(k,y) + (-1)^k\varphi(k,-y) = -B_k(\{y\})\frac{(2\pi i)^k}{k!} \qquad (W = \{\mathrm{id},\sigma_{\alpha_1}\})$$

§4. Special Zeta-Values (at Positive Even Integers)

Functional Relation:

$$\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Big)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};X_r)=(-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Big).$$

Theorem 2. (KMT 2008) *For* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k}=\mathbf{k}$ *for all* $w\in W$,

$$\zeta_r(\mathbf{k},\mathbf{0};X_r)=\frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k},\mathbf{0};X_r)\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Big)\in\mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}.$$

cf. $(X_r=A_1)$

$$\zeta(k)=\frac{-1}{2}B_k\frac{(2\pi i)^k}{k!}\in\mathbb{Q}\pi^k\qquad(k\in 2\mathbb{Z}_{\geq 1})$$

In particular, if $\mathbf{k}=(k)_{\alpha\in\Delta_+}$ with $k\in 2\mathbb{Z}_{\geq 1}$ (that is, all $k_\alpha=k$) satisfies the condition, then $\zeta_r(\mathbf{k},\mathbf{0};X_r)$ coincides with the Witten zeta-function. We also have

$$\begin{aligned}\zeta_2((2,4,4,2),\mathbf{0};C_2)&=\sum_{m,n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2}\\&=\frac{(-1)^4}{2^2 2!}\frac{53}{1513512000}\Big(\frac{(2\pi i)^2}{2!}\Big)^2\Big(\frac{(2\pi i)^4}{4!}\Big)^2=\frac{53}{6810804000}\pi^{12}.\end{aligned}$$

$$\zeta_2((4,8,8,4),\mathbf{0};C_2)=\frac{4611163}{8161801645692895200000}\pi^{24}.$$

§5. Multiple Periodic Bernoulli Functions

$\mathcal{V}$: the set of all bases $\mathbf{V} \subset \Delta_+$.

$\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$: the dual basis of $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta\in\mathbf{V}}$.

$Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ (coroot lattice),

$L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}} \mathbb{Z}\beta^\vee$ ($\Rightarrow |Q^\vee/L(\mathbf{V}^\vee)| < \infty$).

Fix a certain $\phi \in V$ and

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases} \{\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle > 0), \\ 1 - \{-\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle < 0) \end{cases}$$

(a multiple generalization of fractional part).

Definition 2 (generating function, KMT 2010)**.** For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$F(\mathbf{t}, \mathbf{y}; X_r) = \sum_{\mathbf{V}\in\mathcal{V}} \Bigg(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle} \Bigg) \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \Bigg(\prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta \{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \Bigg).$$

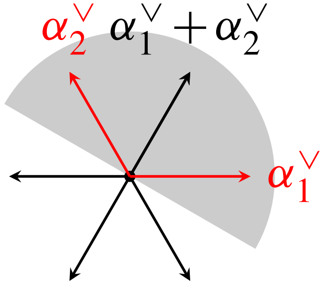
Definition 3 (multiple periodic Bernoulli function, KMT 2008)**.**

$$F(\mathbf{t}, \mathbf{y}; X_r) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k}, \mathbf{y}; X_r) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

cf. ($X_r = A_1$)

$$F(t, y) = \frac{t e^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!}.$$

§6. Example: A_2 Case



$\mathbf{t}=(t_{\alpha_1},t_{\alpha_2},t_{\alpha_1+\alpha_2})=(t_1,t_2,t_3)$,

$\mathbf{y}=y_1\alpha_1^\vee+y_2\alpha_2^\vee$.

$$F(\mathbf{t},\mathbf{y};A_2)=\frac{t_3}{t_3-t_1-t_2}\frac{t_1e^{t_1\{y_1\}}}{e^{t_1}-1}\frac{t_2e^{t_2\{y_2\}}}{e^{t_2}-1}+\frac{t_2}{t_2+t_1-t_3}\frac{t_1e^{t_1\{y_1-y_2\}}}{e^{t_1}-1}\frac{t_3e^{t_3\{y_2\}}}{e^{t_3}-1}+\frac{t_1}{t_1+t_2-t_3}\frac{t_2e^{t_2(1-\{y_1-y_2\})}}{e^{t_2}-1}\frac{t_3e^{t_3\{y_1\}}}{e^{t_3}-1}$$

For $\mathbf{k}=\mathbf{2}=(2,2,2)$,

$$P(\mathbf{2},(y_1,y_2);A_2)=\frac{1}{3780}+\frac{1}{90}(\{y_1\}-\{y_1-y_2\}-\{y_2\})$$
$$\cdots\qquad -\{y_2\}^6-4\{y_1-y_2\}\{y_2\}^5-5\{y_1-y_2\}^2\{y_2\}^4)$$

In particular, $(y_1,y_2)=(0,0)$,

$$\zeta_2(\mathbf{2},(0,0);A_2)=\sum_{m,n=1}^{\infty}\frac{1}{m^2n^2(m+n)^2}=\frac{1}{6}(-1)^3\frac{1}{3780}\frac{(2\pi i)^6}{(2!)^3}=\frac{\pi^6}{2835}.$$

cf. $(X_r=A_1)$

$$F(t,y)=\frac{te^{t\{y\}}}{e^t-1}=\sum_{k=0}^{\infty}B_k(\{y\})\frac{t^k}{k!},\qquad B_2(\{y\})=\frac{1}{6}-\{y\}+\{y\}^2,\qquad \zeta(2)=\frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!}=\frac{\pi^2}{6}.$$

§7. Functional Relations and Special Zeta-Values

Δ_I: the subroot system of Δ with the fundamental roots $\{\alpha_i\}_{i\in I}$ ($I \subset \{1,\ldots,n\}$).

W^I: the minimal coset representatives of the Weyl group W_I of Δ_I (i.e. $W = W_I W^I$).

Theorem 1 (again). *For* $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

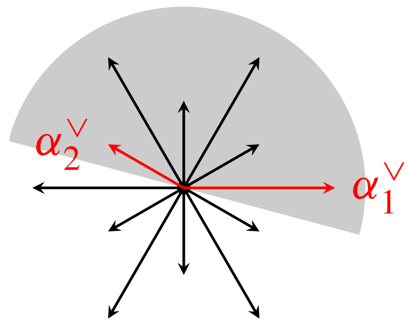
$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; X_r) = (-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr).$$

Theorem 3 (functional relations). *For* $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$ *with* $s_\alpha = k_\alpha \in \mathbb{Z}_{\geq 2}$ *(*$\alpha \in \Delta_+ \setminus \Delta_{I+}$*), we have*

$$\sum_{w\in W^I}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{s}, w^{-1}\mathbf{y}; X_r) = \text{“}\textit{sum of several lower depth zeta-functions}\text{”}.$$

eg. G_2 case. $|W| = 12$ and $|W^{\{2\}}| = 6$.

$$\zeta_2(\mathbf{s},\mathbf{0};G_2) = \sum_{m,n=1}^{\infty}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}(m+3n)^{s_5}(2m+3n)^{s_6}}.$$



$$\zeta_2(3,s,2,2,2,2;G_2) - \zeta_2(3,2,s,2,2,2;G_2) - \zeta_2(2,2,2,s,3,2;G_2)$$
$$- \zeta_2(2,2,2,s,2,3;G_2) - \zeta_2(2,s,2,2,3,2;G_2) - \zeta_2(2,2,s,2,2,3;G_2) = -\frac{958}{243}\zeta(2)\zeta(s+9) + \frac{18917}{2916}\zeta(s+11).$$

$$\zeta(2,4,4,3,3,3;G_2) = \frac{1}{8}\zeta(4)\zeta(15) + \frac{281221}{23328}\zeta(2)\zeta(17) - \frac{11177971}{559872}\zeta(19)$$

§8. Zeta-Functions of Lie Groups

Witten zeta-functions were originally introduced for compact semisimple Lie groups.

Complex semisimple Lie algebra $\mathfrak{g} \Longleftrightarrow$ Simply connected compact semisimple Lie group $\tilde{G}$.

Let L be a lattice satisfying $Q \subset L \subset P$ (Analytically integral forms for a maximal torus of G. ($G \Leftrightarrow L$))

Definition 4 (Zeta-functions of Lie groups)**.** For a root system Δ of type X_r, define

$$\zeta_r(\mathbf{s}, \mathbf{y}; L; X_r) = \sum_{\lambda \in L \cap P_+} e^{2\pi i \langle \mathbf{y}, \lambda + \rho \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda + \rho \rangle^{s_\alpha}}$$

Note $\zeta_r(\mathbf{s}, \mathbf{y}; P; X_r) = \zeta_r(\mathbf{s}, \mathbf{y}; X_r)$ if $G = \tilde{G}$ (simply connected). Let $\pi_1(G)$ be the fundamental group of G.

Theorem 4.

$$\zeta_r(\mathbf{s}, \mathbf{y}; L; X_r) = \frac{1}{|\pi_1(G)|} \sum_{\mu \in L^* / Q^\vee (\simeq \pi_1(G))} (-1)^{\langle \mu, 2\rho \rangle} \zeta_r(\mathbf{s}, \mathbf{y} + \mu; X_r).$$

(weighted average over the universal covering.)

cf. ($X_r = A_1$) with $L = Q$.

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i (2m+1) y}}{(2m+1)^s} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{e^{2\pi i (m+1) y}}{(m+1)^s} + \frac{1}{2} \sum_{m=0}^{\infty} (-1) \frac{e^{2\pi i (m+1)(y + \frac{1}{2})}}{(m+1)^s}$$

$$\zeta(s, y; Q; A_1) = \frac{1}{2} \zeta(s, y; A_1) \qquad + \frac{-1}{2} \zeta(s, y + \frac{1}{2}; A_1).$$

Zeta-functions of Lie groups $=$ L-functions of root systems.

§9. Special Zeta-Values

Value Relation:

$$\zeta_r(\mathbf{s},\mathbf{y};L;X_r)=\frac{1}{|\pi_1(G)|}\sum_{\mu\in L^*/Q^\vee}(-1)^{\langle\mu,2\rho\rangle}\zeta_r(\mathbf{s},\mathbf{y}+\mu;X_r).$$

Definition 5. (multiple periodic Bernoulli function)

$$F(\mathbf{t},\mathbf{y};L;X_r)=\frac{1}{|\pi_1(G)|}\sum_{\mu\in L^*/Q^\vee}(-1)^{\langle\mu,2\rho\rangle}F(\mathbf{t},\mathbf{y}+\mu;X_r)$$

$$=\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}}P(\mathbf{k},\mathbf{y};L;X_r)\prod_{\alpha\in\Delta_+}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

Theorem 5. *For* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k}=\mathbf{k}$ *for all* $w\in W$ *with* $\nu\in P^\vee/Q^\vee=Z(G)$,

$$\zeta_r(\mathbf{k},\nu;L;X_r)=\frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k},\nu;L;X_r)\left(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\right)\in\mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}.$$

In the case $X_r=A_2$ with $L=Q$,

$$\zeta_2(\mathbf{2},\mathbf{0};Q;A_2)=\sum_{\substack{m,n=1\\ m\equiv n \pmod 3}}^{\infty}\frac{1}{m^2n^2(m+n)^2}=\frac{(-1)^3}{3!}\frac{187}{918540}\left(\frac{(2\pi i)^2}{2!}\right)^3=\frac{187\pi^6}{688905}.$$

§§9.1. Example: A_2 Case

There are two cases in the A_2 case:

$$\zeta((s_1,s_2,s_3),\mathbf{y};SU(3)) = \zeta((s_1,s_2,s_3),\mathbf{y};P;A_2) \sum_{m,n=1}^{\infty} \frac{e^{2\pi i\langle \mathbf{y}, m\lambda_1+n\lambda_2\rangle}}{m^{s_1}n^{s_2}(m+n)^{s_3}}.$$

$$\zeta((s_1,s_2,s_3),\mathbf{y};PU(3)) = \zeta((s_1,s_2,s_3),\mathbf{y};Q;A_2) = \sum_{\substack{m,n=1 \\ m\equiv n \pmod 3}}^{\infty} \frac{e^{2\pi i\langle \mathbf{y}, m\lambda_1+n\lambda_2\rangle}}{m^{s_1}n^{s_2}(m+n)^{s_3}}.$$

$$\zeta(\mathbf{4},\lambda_1^\vee;SU(3)) = \sum_{m,n=1}^{\infty} \frac{\varrho^{2m+n}}{m^4n^4(m+n)^4} = \frac{1078771}{16158662895375}\pi^{12} \qquad (\varrho = e^{2\pi i/3}),$$

$$\zeta(\mathbf{6},\lambda_1^\vee;SU(3)) = \frac{88392335894}{503379257192850576037 5}\pi^{18},$$

$$\zeta(\mathbf{8},\lambda_1^\vee;SU(3)) = \frac{1012923518531597}{221554200140503797221045015625}\pi^{24}.$$

$$\zeta(\mathbf{4},\mathbf{0};PU(3)) = \sum_{\substack{m,n=1 \\ m\equiv n \pmod 3}}^{\infty} \frac{1}{m^4n^4(m+n)^4} = \frac{3279473}{48475988686125}\pi^{12},$$

$$\zeta(\mathbf{6},\mathbf{0};PU(3)) = \frac{53109402098}{302027554315710345 6225}\pi^{18},$$

$$\zeta(\mathbf{8},\mathbf{0};PU(3)) = \frac{178778564412743}{3909780002479478774489029687 5}\pi^{24}.$$

§10. Relation with EZ Zeta-Functions

Definition 6 (Euler-Zagier zeta-functions)**.**

$$\zeta_{EZ,r}(s_1,s_2,\ldots,s_r)=\sum_{m_1>m_2>\cdots>m_r>0}\frac{1}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}}.$$

Recall zeta-functions of type C_2, C_3:

$$\zeta(\mathbf{s};C_2)=\sum_{l,m=1}^{\infty}\frac{1}{(l+m)^{s_1}l^{s_2}}\cdot\frac{1}{m^{s_3}(2l+m)^{s_4}},$$

$$\zeta(\mathbf{s};C_3)=\sum_{l,m,n=1}^{\infty}\frac{1}{(l+m+n)^{s_1}(l+m)^{s_2}l^{s_3}}$$

$$\cdot\frac{1}{m^{s_4}n^{s_5}(m+n)^{s_6}(m+2n)^{s_7}(l+m+2n)^{s_8}(l+2m+2n)^{s_9}}.$$

Put $s_\alpha=0$ for irrelevant variables s_α. $\Rightarrow$

$$\zeta(\mathbf{s};C_2)=\sum_{l,m=1}^{\infty}\frac{1}{(l+m)^{s_1}l^{s_2}}=\zeta_{EZ,2}(s_1,s_2),$$

$$\zeta(\mathbf{s};C_3)=\sum_{l,m,n=1}^{\infty}\frac{1}{(l+m+n)^{s_1}(l+m)^{s_2}l^{s_3}}=\zeta_{EZ,3}(s_1,s_2,s_3).$$

Zeta-functions of root systems:

$$\zeta(\mathbf{s},\mathbf{y};\Delta)=\sum_{\lambda\in P_+} e^{2\pi i\langle \mathbf{y},\lambda+\rho\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda+\rho\rangle^{s_\alpha}}.$$

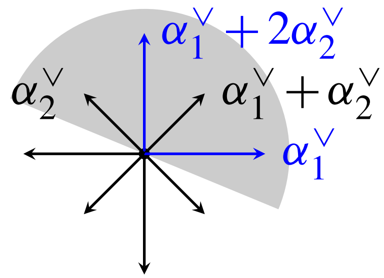
The set of positive coroots in the root system of type C_r:

$$\Delta_+^\vee=(\Delta_+^\vee)^{\rm long}\cup(\Delta_+^\vee)^{\rm short}$$

$$(\Delta_+^\vee)^{\rm short}=\{\alpha_1^\vee+\cdots+\alpha_r^\vee,\quad \alpha_2^\vee+\cdots+\alpha_r^\vee,\quad \ldots,\quad \alpha_{r-1}^\vee+\alpha_r^\vee,\quad \alpha_r^\vee\}.$$

If we put $s_\alpha=0$ for long coroots $\alpha^\vee$, then

$$\zeta(\mathbf{s},\mathbf{0};\Delta)=\sum_{m_1,m_2,\cdots,m_r=1}^{\infty}\frac{1}{(m_1+\cdots+m_r)^{s_1}(m_2+\cdots+m_r)^{s_2}\cdots m_r^{s_r}}.$$



Generally, EZ zeta-functions are realized in any root systems.

In the root system of type C_r, the decomposition is consistent with the action of the Weyl group.

$$W(\Delta_+^\vee)^{\rm long}=(\Delta_+^\vee)^{\rm long},$$

$$W(\Delta_+^\vee)^{\rm short}=(\Delta_+^\vee)^{\rm short}.$$

In the root system of type C_r, the decomposition is consistent with the action of the Weyl group.

$$W(\Delta_+^\vee)^{\mathrm{long}} = (\Delta_+^\vee)^{\mathrm{long}},$$

$$W(\Delta_+^\vee)^{\mathrm{short}} = (\Delta_+^\vee)^{\mathrm{short}}.$$

Theorem 1 (again). *For* $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; X_r) = (-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr).$$

Theorem 2 (again). *For* $\mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in (2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k} = \mathbf{k}$ *for all* $w \in W$,

$$\zeta_r(\mathbf{k},\mathbf{0};X_r) = \frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k},\mathbf{0};X_r)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr) \in \mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}.$$

We want to put $s_\alpha = 0$.

$\Downarrow$

Theorems can not be applied.

(In fact, a formal substitution gives a wrong result.)

Put $\mathbf{y}=\sum_{i=1}^{r} y_i\alpha_i^\vee$. For $\alpha\in\Delta_+$ ($\alpha^\vee=\sum_{i=1}^{r} a_i\alpha_i^\vee$), define $\mathcal{D}_\alpha$ (elimination of a root α) by

$$\mathcal{D}_\alpha=\frac{\partial}{\partial t_\alpha}\Big|_{t_\alpha=0}\left(\sum_{i=1}^{r} a_i\frac{\partial}{\partial y_i}\right).$$

Definition 7. For $A\subset\Delta_+$, put $\mathbf{t}_A=\{t_\alpha\}_{\alpha\in\Delta\setminus A}$.

$$\begin{aligned}F_A(\mathbf{t}_A,\mathbf{y};\Delta)&=\Bigl(\Bigl(\prod_{\alpha\in A}\mathcal{D}_\alpha\Bigr)F\Bigr)(\mathbf{t},\mathbf{y};\Delta)\\&=\sum_{\mathbf{k}_A\in\mathbb{Z}_{\geq 0}^{|\Delta_+\setminus A|}} P_A(\mathbf{k}_A,\mathbf{y};\Delta)\prod_{\alpha\in\Delta_+\setminus A}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}.\end{aligned}$$

Theorem 6. *For $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}$, $k_\alpha\in\mathbb{Z}_{\geq 1}$ ($\alpha\in\Delta_+\setminus A$), $k_\alpha=0$ ($\alpha\in A$), we have*

$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta(w^{-1}\mathbf{k},w^{-1}\mathbf{y};\Delta)=(-1)^{|\Delta_+|}P_A(\mathbf{k}_A,\mathbf{y};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr).$$

cf. $\mathcal{D}_\alpha$ is the lift of the following to the level of the generating function.

$$\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)\sum_{m,n=1}^{\infty}\frac{e^{2\pi i(my_1+ny_2)}}{m^{k_1}n^{k_2}(m+n)}=2\pi i\sum_{m,n=1}^{\infty}\frac{e^{2\pi i(my_1+ny_2)}}{m^{k_1}n^{k_2}}.$$

Theorem 7 (generating function for some $s_\alpha = 0$). *For $A = \{\nu_1, \ldots, \nu_N\} \subset \Delta_+$,*

$$\Big(\mathcal{D}_{\nu_N} \cdots \mathcal{D}_{\nu_1} F\Big)(\mathbf{t}, \mathbf{y}; \Delta) = \sum_{\mathbf{V} \in \mathcal{V}_A} (-1)^{|A \setminus \mathbf{V}|} \Big(\prod_{\gamma \in \Delta_+ \setminus (\mathbf{V} \cup A)} \frac{t_\gamma}{t_\gamma - \sum_{\beta \in \mathbf{V} \setminus A} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \Big)$$
$$\times \frac{1}{|Q^\vee / L(\mathbf{V}^\vee)|} \sum_{q \in Q^\vee / L(\mathbf{V}^\vee)} \Big(\prod_{\beta \in \mathbf{V} \setminus A} \frac{t_\beta \exp(t_\beta \{\mathbf{y} + q\}_{\mathbf{V}, \beta})}{e^{t_\beta} - 1} \Big),$$

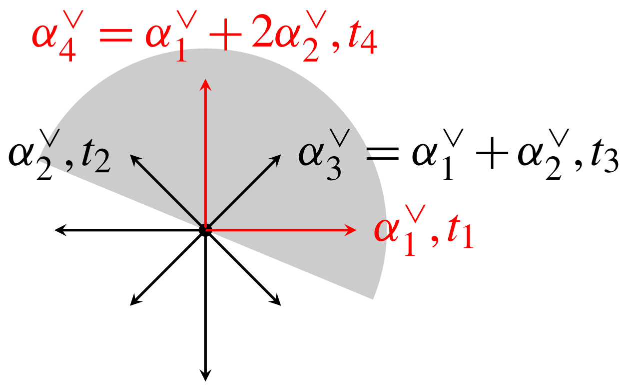
where $\langle B \rangle$ denotes the linear hull of B and

$$\mathcal{V}_A = \{\mathbf{V} \in \mathcal{V} \mid \nu_{j+1} \notin \langle \mathbf{V} \cap A_j \rangle \text{ for all } j \in \{0, \ldots, N-1\}\}.$$

eg. In the root system of type C_2, put $k_\alpha = 0$ for long coroots $\alpha^\vee$.

$\mathbf{y} = y_1 \alpha_1^\vee + y_2 \alpha_2^\vee$.

$$\mathcal{D}_{\alpha_1} = \frac{\partial}{\partial t_1}\Big|_{t_1 = 0} \Big(\frac{\partial}{\partial y_1}\Big),$$
$$\mathcal{D}_{\alpha_4} = \frac{\partial}{\partial t_4}\Big|_{t_4 = 0} \Big(\frac{\partial}{\partial y_1} + 2\frac{\partial}{\partial y_2}\Big).$$



$$(\mathcal{D}_{\alpha_1} \mathcal{D}_{\alpha_4} F)(t_1, t_2, t_3, t_4, y_1, y_2; C_2) = F_{EZ,2}(t_2, t_3, y_1, y_2).$$

$$F_{EZ,2}(t_2,t_3,y_1,y_2) = 1 + \frac{t_2t_3e^{\{y_2\}t_2}}{(e^{t_2}-1)(t_2-t_3)} + \frac{t_2t_3e^{\{y_2\}t_3}}{(e^{t_3}-1)(-t_2+t_3)} + \frac{t_2t_3e^{(1-\{y_1-y_2\})t_2+\{y_1\}t_3}}{(e^{t_2}-1)(e^{t_3}-1)}$$
$$- \frac{t_2t_3e^{(1-\{2y_1-y_2\})t_2}}{(e^{t_2}-1)(t_2+t_3)} - \frac{t_2t_3e^{\{2y_1-y_2\}t_3}}{(e^{t_3}-1)(t_2+t_3)}$$
$$= \sum_{k_2,k_3=1}^{\infty} P(k_2,k_3,y_1,y_2)\frac{t_2^{k_2}t_3^{k_3}}{k_2!k_3!}.$$

Special values:

$$\zeta_{EZ,2}(4,4) = \frac{(-1)^4}{8}P(4,4,0,0)\frac{(2\pi i)^{4+4}}{4!4!} = \frac{(-1)^4}{8}\frac{1}{6300}\frac{(2\pi i)^{4+4}}{4!4!} = \frac{\pi^8}{113400}.$$

Value relations: (A refinement of restricted sum formula)

$$\zeta_{EZ,2}(4,2)+\zeta_{EZ,2}(2,4) = \frac{(-1)^4}{4}P(4,2,0,0)\frac{(2\pi i)^{4+2}}{4!2!} = \frac{(-1)^4}{4}\left(-\frac{1}{420}\right)\frac{(2\pi i)^{4+2}}{4!2!} = \frac{\pi^6}{1260},$$

$$\zeta_{EZ,2}(6,4)+\zeta_{EZ,2}(4,6) = \frac{(-1)^4}{4}P(6,4,0,0)\frac{(2\pi i)^{6+4}}{6!4!} = \frac{(-1)^4}{4}\left(-\frac{1}{13860}\right)\frac{(2\pi i)^{6+4}}{6!4!} = \frac{\pi^{10}}{935550}.$$

cf. sum formula

$$\sum_{\substack{k_1+\cdots+k_n=k \\ k_i\geq 1, k_1\geq 2}} \zeta_{EZ,n}(k_1,\ldots,k_n) = \zeta(k).$$

eg. In the root system of type C_3, put $k_\alpha = 0$ for long coroots $\alpha^\vee$.

Special values:

$$\zeta_{EZ,3}(2,2,2) = \frac{(-1)^9}{48} P(2,2,2,0,0,0) \frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{(-1)^9}{48} \frac{1}{840} \frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{\pi^6}{5040}.$$

Value relations: (A refinement of restricted sum formula)

$$\begin{aligned} &\zeta_{EZ,3}(2,2,4) + \zeta_{EZ,3}(2,4,2) + \zeta_{EZ,3}(4,2,2) \\ &\quad = \frac{(-1)^9}{16} P(2,2,4,0,0,0) \frac{(2\pi i)^{2+2+4}}{2!2!4!} = \frac{(-1)^9}{16} \left(-\frac{1}{7560}\right) \frac{(2\pi i)^{2+2+4}}{2!2!4!} = \frac{\pi^8}{45360}, \end{aligned}$$

$$\begin{aligned} &\zeta_{EZ,3}(2,4,6) + \zeta_{EZ,3}(2,6,4) + \zeta_{EZ,3}(4,2,6) + \zeta_{EZ,3}(4,6,2) + \zeta_{EZ,3}(6,2,4) + \zeta_{EZ,3}(6,4,2) \\ &\quad = \frac{(-1)^9}{8} P(2,4,6,0,0,0) \frac{(2\pi i)^{2+4+6}}{2!4!6!} = \frac{(-1)^9}{8} \left(-\frac{1}{315315}\right) \frac{(2\pi i)^{2+4+6}}{2!4!6!} = \frac{2\pi^{12}}{42567525}. \end{aligned}$$

Odd cases can be treated in this framework.

§11. Generalizations of EZ Zeta-Functions

$$\zeta_{EZ,r}(s_1,s_2,\ldots,s_r) \Longleftrightarrow \zeta(\mathbf{s},\mathbf{0};C_r) \text{ with } s_\alpha = 0 \text{ for long coroots } \alpha^\vee.$$

Thus we can consider the following generalizations. (there are at most two lengthes. $X_n = B_n, C_n, F_4, G_2$)

1. $\zeta(\mathbf{s},\mathbf{0};X_r)$ with $s_\alpha = 0$ for long coroots $\alpha^\vee$. (C_r, EZ Zeta-functions)
2. $\zeta(\mathbf{s},\mathbf{0};X_r)$ with $s_\alpha = 0$ for short coroots $\alpha^\vee$.
3. $\zeta(\mathbf{s},\mathbf{0};X_r)$ itself.

1. $\displaystyle \zeta(\mathbf{s};B_2) = \sum_{l,m=1}^{\infty} \frac{1}{l^{s_1}(l+m)^{s_2}},$

   $\displaystyle \zeta(\mathbf{s};B_3) = \sum_{l,m,n=1}^{\infty} \frac{1}{l^{s_1}m^{s_2}(l+m)^{s_3}(m+n)^{s_4}(l+m+n)^{s_5}(l+2m+n)^{s_6}}.$

2. $\displaystyle \zeta(\mathbf{s};B_2) = \sum_{l,m=1}^{\infty} \frac{1}{m^{s_1}(2l+m)^{s_2}}, \qquad \zeta(\mathbf{s};B_3) = \sum_{l,m,n=1}^{\infty} \frac{1}{n^{s_1}(2m+n)^{s_2}(2l+2m+n)^{s_3}}.$

   $\displaystyle \zeta(2,2,2;B_3) = \sum_{l,m,n=1}^{\infty} \frac{1}{n^2(2m+n)^2(2l+2m+n)^2} = \frac{\pi^6}{40320}.$

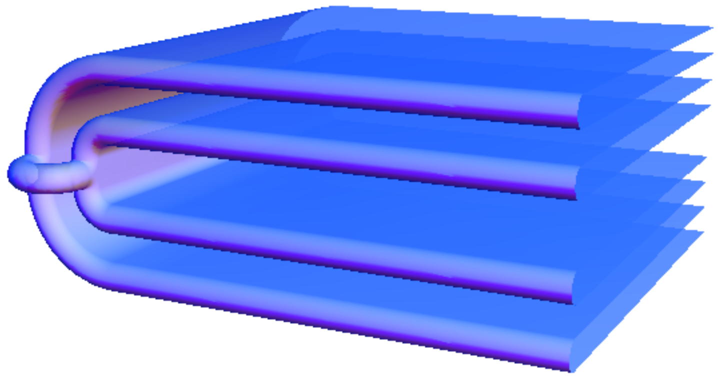
$$\zeta(G_2) = \sum_{m_1,m_2=1}^{\infty} \frac{1}{m_1(m_1+3m_2)(2m_1+3m_2)} \frac{1}{m_2(m_1+m_2)(m_1+2m_2)}$$

$$\zeta(F_4) = \sum_{m_1,m_2,m_3,m_4=1}^{\infty} \frac{1}{m_1 m_2(m_1+m_2)(m_2+2m_3)}$$
$$\frac{1}{(m_2+2m_3+2m_4)(m_1+m_2+2m_3)(m_1+2m_2+2m_3)}$$
$$\frac{1}{(m_1+m_2+2m_3+2m_4)(m_1+2m_2+2m_3+2m_4)(m_1+2m_2+4m_3+2m_4)}$$
$$\frac{1}{(m_1+3m_2+4m_3+2m_4)(2m_1+3m_2+4m_3+2m_4)}$$
$$\frac{1}{m_3 m_4(m_2+m_3)(m_3+m_4)}$$
$$\frac{1}{(m_1+m_2+m_3)(m_2+m_3+m_4)(m_2+2m_3+m_4)}$$
$$\frac{1}{(m_1+m_2+m_3+m_4)(m_1+m_2+2m_3+m_4)(m_1+2m_2+2m_3+m_4)}$$
$$\frac{1}{(m_1+2m_2+3m_3+m_4)(m_1+2m_2+3m_3+2m_4)}$$

Details are future work.

§12. Integral Representation

Theorem 8 (K. 2006, 2007)**.**



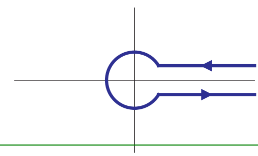
Σ : *a union of smooth surfaces.*

S : *a set of linear functionals on* $\mathbb{C}^N$.

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_R=0}^{\infty}\frac{e^{\xi_1 m_1}\cdots e^{\xi_R m_R}}{(a_1+b_{11}m_1+\cdots+b_{1R}m_R)^{s_1}\cdots(a_N+b_{N1}m_1+\cdots+b_{NR}m_R)^{s_N}}=$$

$$\frac{1}{\Gamma(s_1)\cdots\Gamma(s_N)}\prod_{t\in S}\frac{1}{e^{2\pi i t(s)}-1}\int_{\Sigma}\frac{e^{(b_{11}+\cdots+b_{1R}-a_1)z_1}\cdots e^{(b_{N1}+\cdots+b_{NR}-a_N)z_N}z_1^{s_1-1}\cdots z_N^{s_N-1}}{(e^{z_1 b_{11}+\cdots+z_N b_{N1}}-e^{\xi_1})\cdots(e^{z_1 b_{1R}+\cdots+z_N b_{NR}}-e^{\xi_R})}dz_1\wedge\cdots\wedge dz_N.$$

cf. $$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{\Gamma(s)(e^{2\pi i s}-1)}\int_C\frac{z^{s-1}}{e^z-1}dz.$$ C: Hankel contour



Theorem 9.

$$\zeta_r(\mathbf{s},\mathbf{y};X_r)=\sum_{\lambda\in P_+}e^{2\pi i\langle \mathbf{y},\lambda+\rho\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda+\rho\rangle^{s_\alpha}}$$

$$=\prod_{\alpha\in\Delta_+}\frac{1}{\Gamma(s_\alpha)}\prod_{t\in S_\Delta}\frac{1}{e^{2\pi i t(\mathbf{s})}-1}\int_{\Sigma_\Delta}\frac{\prod_{\alpha\in\Delta_+}z_\alpha^{s_\alpha-1}}{\prod_{j=1}^{r}(e^{\sum_{\alpha\in\Delta_+}z_\alpha\langle\alpha,\lambda_j\rangle}-e^{2\pi i\langle \mathbf{y},\lambda_j\rangle})}\bigwedge_{\alpha\in\Delta_+}dz_\alpha.$$

§§12.1. Example: A_2 case

$$
\begin{aligned}
\zeta((s_1,s_2,s_3),\mathbf{0};A_2) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}} \\
&= \left.\begin{aligned}&\frac{1}{\Gamma(s_1)\Gamma(s_2)\Gamma(s_3)(e^{2\pi i s_1}-1)(e^{2\pi i s_2}-1)(e^{2\pi i s_3}-1)} \\
&\quad \times \frac{1}{(e^{2\pi i(s_1+s_3)}-1)(e^{2\pi i(s_2+s_3)}-1)(e^{2\pi i(s_1+s_2+s_3)}-1)}\end{aligned}\right\} \qquad \text{(Singular)} \\
&\quad \times \int_{\Sigma}\frac{z_1^{s_1-1}z_2^{s_2-1}z_3^{s_3-1}}{(e^{z_1+z_3}-1)(e^{z_2+z_3}-1)}dz_1\wedge dz_2\wedge dz_3. \qquad \text{(Holomorphic).}
\end{aligned}
$$

$$
\begin{aligned}
\zeta((\overset{1}{0},\overset{2}{0},\overset{3}{0}),\mathbf{0};A_2) &= \zeta((\overset{2}{0},\overset{1}{0},\overset{3}{0}),\mathbf{0};A_2) = \frac{5}{12}, \\
\zeta((\overset{1}{0},\overset{3}{0},\overset{2}{0}),\mathbf{0};A_2) &= \zeta((\overset{3}{0},\overset{1}{0},\overset{2}{0}),\mathbf{0};A_2) = \frac{1}{3}, \\
\zeta((\overset{2}{0},\overset{3}{0},\overset{1}{0}),\mathbf{0};A_2) &= \zeta((\overset{3}{0},\overset{2}{0},\overset{1}{0}),\mathbf{0};A_2) = \frac{1}{4},
\end{aligned}
$$

where

$$
\zeta((\overset{1}{0},\overset{2}{0},\overset{3}{0}),\mathbf{0};A_2)\ \lim_{s_3\to 0}\lim_{s_2\to 0}\lim_{s_1\to 0}\zeta((s_1,s_2,s_3),\mathbf{0};A_2),
$$

etc.